The Hard Lefschetz Theorem and the topology of semismall maps

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Abstract

We introduce the notion of *lef* line bundles on a complex projective manifold. We prove that lef line bundles satisfy the Hard Lefschetz Theorem, the Lefschetz Decomposition and the Hodge-Riemann Bilinear Relations. We study proper holomorphic semismall maps from complex manifolds and prove that, for constant coefficients, the Decomposition Theorem is equivalent to the non-degeneracy of certain intersection forms. We give a proof of the Decomposition Theorem for the complex direct image of the constant sheaf when the domain and the target are projective by proving that the forms in question are non-degenerate. A new feature uncovered by our proof is that the forms are definite.

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1 Introduction

Let X be a complex projective n-fold endowed with a line bundle L. The classical statement of the Hard Lefschetz Theorem is that if L is ample then, for every $r \geq 0$, cupping with $c_1(L)^r$ gives an isomorphism of $H^{n-r}(X, \mathbb{Q})$ onto $H^{n+r}(X, \mathbb{Q})$.

In this paper we introduce the notion of *lef* line bundles on projective manifolds. They are precisely the line bundles with the property that $L^{\otimes t} = f^*A$, where $f : X \to Y$ is a projective semismall morphism, t is some positive integer and A is an ample line bundle. Recall that f is *semismall* when the set of points in Y, whose fiber has dimension k, has codimension at least $2k$. One has the following strict implications: ample $\Rightarrow$ lef $\Rightarrow$ semiample+big; the results we prove for lef line bundles fail for the last class.

Our first result is Theorem 2.3.1: if L is a line bundle such that a positive power is generated by its global sections, then the Hard Lefschetz Theorem holds for L *if and only if* L is lef. See also Proposition 2.2.7. A Lefschetz-type decomposition for $H^*(X, \mathbb{Q})$ ensues and we prove the Hodge-Riemann Bilinear relations, i.e. the signature of the intersection form on the associated primitive spaces has properties completely analogous to the ample case. We offer two proofs. One is by induction on hyperlane sections; see §2.3.1. Another one is by the use of pencils; see §2.3.3. While the first is simpler, the second one characterizes monodromy invariants.

A second result is Theorem 2.4.1 where we prove a higher dimensional generalization of the Grauert criterion for the contractibility of a finite set of curves on a surface: if $f : X \to Y$ is a semismall map between $2m$-dimensional projective varieties, X is nonsingular, $y \in Y$ is a point and $\dim f^{-1}(y) = m$, then the intersection form associated with the irreducible components of maximal dimension of $f^{-1}(y)$ is $(-1)^m$-definite. We call this property the *Hodge Index Theorem for semismall maps*. It is proved using Theorem 2.3.1, the characterization of monodromy invariants and Goresky-MacPherson version of the Weak Lefschetz Theorem.

This result has a surprising consequence when applied to study the topology of semismall maps.
Beilinson-Bernstein-Deligne, [1], proved the following fundamental result, called the Decomposition Theorem: if $g : U \to V$ is a proper morphism of algebraic varieties, then the direct image complex of an intersection cohomology complex of geometric origin on U, e.g. the intersection cohomology complex of U, is isomorphic, in the derived category, to a direct sum of *shifted* intersection cohomology complexes. An analogous result holds over any field. Even over the complex numbers, finite fields are necessary to their proof. Saito, [17], proved and extended these results in a Kähler setting using mixed Hodge modules. Borho-MacPherson, [3], have observed that, in the statement of the Decomposition Theorem, the shifts disappear when g is semismall, U is smooth and constant coefficients are taken.

Let $f : X \to Y$ be a proper *semismall* holomorphic map from a complex manifold. We propose a new approach to the Decomposition Theorem for such maps and for constant

coefficients. Note that if f is not Kähler, see [17], then it is not known whether the theorem holds or not. The counterexamples to the Decomposition Theorem we are aware of are not for semismall maps.

We prove Theorem 3.3.3: the Decomposition Theorem is equivalent to the non-degeneracy of a collection of intersection forms that arise by studying a Mather-Thom-Whitney stratification of f. This result holds also for more general (i.e. non complex analytic) maps admitting stratifications with properties analogous to the complex analytic ones.

Using our Hodge Index Theorem for semismall maps we give a new proof of the Decomposition Theorem for constant coefficients and for X and Y projective, Theorem 3.4.1. This was actually our starting point and our main motivation to investigate lef bundles (l.e.f.= *Lefschetz effettivamente funziona*). A remarkable new feature we prove is that the intersection forms in questions are not only non-degenerate, but also definite.

We believe that the relation between the Decomposition Theorem and the intersection forms associated with the strata is illuminating. Furthermore, the direct proof of the non-degeneracy of these forms, with the new additional information on the signatures, neither relying on reduction to positive characteristic nor on Saito's theory of mixed Hodge modules, sheds light on the geometry underlying the decomposition theorem and gives some indications on its possible extensions beyond the algebraic category.

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2 The Hard Lefschetz Theorem for lef line bundles

In this section we introduce the notion of *lef* line bundle on a projective variety. It is a positivity notion weaker than ampleness but stronger than semiampleness and bigness combined. Lef line bundles satisfy many of the cohomological properties of ampleness. Our main result is that they also satisfy the Hard Lefschetz Theorem, the Lefschetz Decomposition and the Hodge Riemann Bilinear relations on the primitive spaces. These results are all false, in general, for line bundles which are simultaneously generated by their global sections and big. We also study pencils of sections and obtain more precise statements and a second proof of the Hard Lefschetz Theorem. This extra information is then used to prove our Hodge Index Theorem for semismall maps.

2.1 Semismall maps and lef line bundles

Let $f : X \to Y$ be a proper holomorphic map. For every integer k define $Y^k := \{y \in Y \mid \dim f^{-1}(y) = k\}$. The spaces Y^k are locally closed analytic subvarieties of Y whose

disjoint union is Y. If a fiber is reducible, then it is understood that its dimension is the highest among the dimensions of its components.

Definition 2.1.1 We say that a proper holomorphic map $f : X \to Y$ of irreducible varieties is *semismall* if $\dim Y^k + 2k \leq \dim X$ for every k. Equivalently, f is semismall if and only if there is no irreducible subvariety $T \subseteq X$ such that $2\dim T - \dim f(T) > \dim X$.

Remark 2.1.2 A semismall map is necessarily generically finite.

From now on we shall assume that semismall maps are proper and surjective.

Definition 2.1.3 We say that a line bundle M on a complex projective variety X is *lef* if a positive multiple of M is generated by its global sections and the corresponding morphism onto the image is semismall.

Remark 2.1.4 If the map associated to a multiple of M generated by its global sections is semismall, then the map associated with any other multiple of M generated by its global sections is semismall as well. A lef line bundle is nef and big, but not conversely.

Proposition 2.1.5 *(Weak Lefschetz theorem for lef line bundles) Let M be a lef line bundle on a smooth complex projective variety X. Assume that M admits a section $s \in H^0(X, M)$ whose reduced zero locus is a smooth divisor Y. Denote by $i : Y \to X$ the inclusion.*
The restriction map $i^* : H^r(X) \longrightarrow H^r(Y)$ is an isomorphism for $r < dimX - 1$ and it is injective for $r = \dim X - 1$.

Proof. The proof can be obtained by a use of the Leray spectral sequence coupled with the theorem on the cohomological dimension of constructible sheaves on affine varieties. See, for example, [16]. See also [9], Lemma 1.2 and Proposition 2.1.6. □

We shall employ the following strenghtening of the Weak Lefschetz Theorem due to Goresky-MacPherson, [12], II.1.1. In fact their statement is even stronger.

Proposition 2.1.6 *Let $f : X \to Y$ be a semismall holomorphic map from a complex connected manifold onto a quasi-projective variety Y, and $H \subseteq Y$ be a general hyperplane section.*
The restriction map $i^* : H^r(X) \longrightarrow H^r(f^{-1}(H))$ is an isomorphism for $r < dimX - 1$ and it is injective for $r = \dim X - 1$.

Proposition 2.1.7 *(Bertini theorem for lef line bundles) Let M be a lef line bundle on a nonsingular complex projective variety X. Assume that M is generated by its global sections. Let $W' \subseteq |M|$ be the set of divisors Y in the linear system of M such that Y is smooth and $M_{|Y}$ is lef. Then the set W' contains a nonempty and Zariski open subset $W \subseteq |M|$.*

Proof. By virtue of the standard Bertini Theorem, a generic divisor $D \in |M|$ is nonsingular. Let $f : X \to Y$ be the semismall map associated with $|M|$ and Y^k be the locally closed subvarieties mentioned above. The set of divisors containing at least one among the closed subvarieties $f^{-1}(\overline{Y^k})$ is a finite union of linear proper subspaces of $|M|$. The conclusion follows. $\square$

Remark 2.1.8 The Kodaira-Akizuki-Nakano Vanishing Theorem holds for lef line bundles. See [9], Theorem 2.4. See also [16].

Remark 2.1.9 The Kodaira-Akizuki-Nakano Vanishing Theorem for lef line bundles coupled with Serre Vanishing Theorem implies that if $f : X \to Y$ is a semismall holomorphic map from a complex projective manifold of dimension n onto a projective variety then $R^q f_* \Omega^p_X = 0$ for every $p + q > n$. This fact generalizes Grauert-Riemannschneider Vanishing Theorem and is false for semiample big line bundles.

Remark 2.1.10 One can define lef vector bundles in terms of the lefness of the tautological line bundle and prove that many of the standard properties for ample vector bundles hold for lef ones.

2.2 The property HL

Let X be a smooth, compact, oriented manifold of even real dimension $2n$. We will use the notation $H^r(X)$ for $H^r(X, \mathbb{Q})$. The bilinear form on $H^*(X) := \oplus H^r(X)$ defined by $(\alpha, \beta) = \int_X \alpha \wedge \beta$ is non-degenerate by Poincaré duality.
Let $\omega \in H^2(X)$. We define a bilinear form on $H^{n-r}(X)$ by setting

$$\Psi(\alpha, \beta) = (-1)^{\frac{(n-r)(n-r-1)}{2}} \int_X \omega^r \wedge \alpha \wedge \beta,$$

for every $0 \leq r \leq n$. The form Ψ is non-degenerate precisely when the linear map $L^r = L^r_\omega : H^{n-r}(X) \longrightarrow H^{n+r}(X)$, sending α to $\omega^r \wedge \alpha$, is an isomorphism.

Definition 2.2.1 We say that (X, ω) *has property HL_r* if the map $L^r_\omega : H^{n-r}(X) \longrightarrow H^{n+r}(X)$ given by the cup product with ω^r is an isomorphism.
We say that (X, ω) has property HL if it has property HL_r for every $0 \leq r \leq n$.

Note that property HL_0 is automatic and that property HL_n is equivalent to $\int_X \omega^n \neq 0$.
Define $H^{n-r}(X) \supseteq P^{n-r} = P^{n-r}_\omega := Ker\, L^{r+1}_\omega$ and call its elements *primitive* (with respect to ω). The following Lefschetz-type decomposition is immediate.

Proposition 2.2.2 *Assume that (X, ω) has property HL. For every $0 \leq r \leq n$, we have the following "primitive" decomposition*

$$H^{n-r}(X) = P^{n-r} \oplus L_\omega(H^{n-r-2}(X)).$$

There is a direct sum decomposition

$$H^{n-r}(X) = \oplus L^i_\omega P^{n-r-2i}.$$

The subspaces $L^i_\omega P^{n-r-2i}$ are pairwise orthogonal in $H^{n-r}(X)$.

Remark 2.2.3 The projection of $H^{n-r}(X)$ onto P^{n-r} is given by $\alpha \to \alpha - L_\omega(L^{r+2}_\omega)^{-1} L^{r+1}_\omega \alpha$, where $(L^{r+2}_\omega)^{-1}$ denotes the inverse to $(L^{r+2}_\omega) : H^{n-r-2}(X) \to H^{n+r+2}(X)$.

Definition 2.2.4 A *rational Hodge structure of pure weight* r is a rational vector space H with a bigraduation of $H_\mathbb{C} = H \otimes \mathbb{C} = \oplus H^{p,q}$ for $p + q = r$ such that $H^{p,q} = \overline{H^{q,p}}$.

Definition 2.2.5 A *polarization* of the weight r Hodge structure H is a bilinear form Ψ on H, symmetric for r even, anti-symmetric for r odd, such that its $\mathbb{C}$-bilinear extension, to $H_\mathbb{C}$, still denoted by Ψ, satisfies :
a) the spaces $H^{p,q}$ and $H^{s,t}$ are Ψ-orthogonal whenever either $p \neq t$, or $q \neq s$;
b) $i^{p-q}\Psi(\alpha, \overline{\alpha}) > 0$, for every non-zero $\alpha \in H^{p,q}$.

Let X be a nonsingular complex projective variety. For any ample line bundle M define $L_M := L_{c_1(M)} : H^{n-r}(X) \longrightarrow H^{n-r+2}(X)$. Classical Hodge theory gives that $\Psi_M(\alpha, \beta) = (-1)^{\frac{(n-r)(n-r-1)}{2}} \int_X L^r_M(\alpha) \wedge \beta$ is a polarization of the weight $(n-r)$ Hodge structure $P^{n-r}_M = Ker L^{r+1}_M \subseteq H^{n-r}(X)$.

We say that (X, M) has property HL_r (HL, resp.) if $(X, c_1(M))$ has property HL_r (HL, resp.).

The HL property for a pair (X, M) with X projective and M nef implies that Ψ_M is a polarization. In fact such a line bundle can be written as a limit of rational Kähler classes and the following proposition applies.

Proposition 2.2.6 *Let X be a compact connected complex Kähler manifold of dimension n and M be a line bundle such that (X, M) has property HL and $c_1(M) = \lim_{i\to\infty} \omega_i$, ω_i Kähler. The bilinear form $\Psi_M(\alpha, \beta) = (-1)^{\frac{(n-r)(n-r-1)}{2}} \int_X L^r_M(\alpha) \wedge \beta$ is a polarization of the weight $(n-r)$ Hodge structure $P^{n-r}_M = Ker L^{r+1}_M \subseteq H^{n-r}(X)$, for every $0 \le r \le n$.*

Proof. Since M is nef we can find rational Kähler classes $\omega_i \in H^2(X)$ such that $\lim_{i\to\infty} \omega_i = c_1(M)$.
The only thing that needs to be proved is the statement $i^{p-q}\Psi_M(\alpha, \overline{\alpha}) > 0$ for every non-zero $\alpha \in P^{n-r}_M \cap H^{p,q}(X)$.
Since the classes ω_i are Kähler, we have the decomposition $H^{n-r}(X) = P^{n-r}_{\omega_i} \oplus L_{\omega_i}(H^{n-r-2}(X))$.
Let π_i denote the projection onto $P^{n-r}_{\omega_i}$,

$$\pi_i(\alpha) = \alpha - L_{\omega_i}(L^{r+2}_{\omega_i})^{-1} L^{r+1}_{\omega_i}\alpha.$$

Since M satisfies the HL condition, the map $L^{r+2}_M : H^{n-r-2}(X) \to H^{n+r+2}(X)$ is invertible so that $\lim_{i\to\infty} (L^{r+2}_{\omega_i})^{-1} = (L^{r+2}_M)^{-1}$. Identical considerations hold for the (p, q)-parts of these invertible maps. It follows that, if $\alpha \in P^{n-r}_M \cap H^{p,q}(X)$, then $\lim_{i\to\infty} \pi_i(\alpha) = \alpha$. Since the operators π_i are of type $(0, 0)$, $\pi_i(\alpha) \in P^{n-r}_{\omega_i} \cap H^{p,q}(X)$. Therefore, $i^{p-q}\Psi_{\omega_i}(\pi_i(\alpha), \overline{\pi_i(\alpha)}) > 0$. It follows that $i^{p-q}\Psi_M(\alpha, \overline{\alpha}) \ge 0$. The HL property for M implies that Ψ_M is non-degenerate, therefore Ψ_M is a polarization of P^{n-r}_M. □

The following elementary fact highlights the connection between the HL property and lef line bundles.

Proposition 2.2.7 *Let $f : X \to Y$ be a surjective projective morphism from a nonsingular projective variety X, A be a line bundle on Y and $M := f^*A$.*
If M has property HL, then f is semismall.

Proof. If f is not semismall, then there exists an irreducible subvariety $T \subseteq X$ such that $2\dim T - n > \dim f(T)$. Let $[T] \in H^{2(n-\dim T)}(X) = H^{n-(2\dim T-n)}(X)$ be the fundamental class of T. The class $c_1(M)^{2\dim T-n}$ can be represented by a $\mathbb{Q}$-algebraic cycle that does not intersect T. It follows that $c_1(M)^{(2\dim T-n)} \cdot [T] = 0$, i.e. M does not satisfy $HL_{2\dim T-n}$. □

Remark 2.2.8 A related fact, dealing with extremal ray contractions, can be found in [18].

2.3 The Hard Lefschetz Theorem and the signature of intersection forms

Our goal is to prove the following extension of the classical Hard Lefschetz Theorem which also constitutes a converse to Proposition 2.2.7. At the same time we prove that the Hodge-Riemann Bilinear Relations hold on the corresponding primitive spaces.

Theorem 2.3.1 *Let X be a nonsingular complex projective variety and M be a lef line bundle on X.*
The pair (X, M) has property HL. In addition, Ψ_M is a polarization of $P^{n-r}_M = Ker\, L^{r+1}_M$.

Remark 2.3.2 Proposition 2.2.2 implies the decomposition of the singular cohomology of X into subspaces which are primitive with respect to M. It is immediate to check that $\dim_{\mathbb{C}} P^l_M = b_l - b_{l-2}$ and $\dim_{\mathbb{C}} P^{p+q}_M \cap H^{p,q}(X) = h^{p,q}(X) - h^{p-1,q-1}(X)$.

The proof of Theorem 2.3.1 can be found in §2.3.1. A second proof, based on a variation of Deligne's proof of the Hard Lefschetz Theorem, see [8], can be found in §2.3.3.

2.3.1 Proof of Theorem 2.3.1

We use induction on the dimension of X. Note that the case $n = 1$ is classical, for M is then necessarily ample. The statement is invariant under taking non-zero positive powers of the line bundle M so that we shall assume from now on that M is generated by its global sections.

In this section X denotes a nonsingular complex projective variety of dimension $n \geq 2$, and M a lef line bundle on X generated by its global sections. Let $s \in H^0(X, M)$ be a section with smooth zero locus Y such that $M_{|Y}$ is lef. Such a section exists by Proposition 2.1.7. Note that Y is necessarily connected by Bertini Theorem. Denote by $i : Y \to X$ the inclusion.

Let $\hat{L}^r_M := (L_{M_{|Y}})^r : H^{n-1-r}(Y) \to H^{n-1+r}(Y)$. We record, for future use, the following elementary fact that follows at once from the projection formula and Poincaré Duality.

Lemma 2.3.3 $L^r_M = i_* \circ \hat{L}^{r-1}_M \circ i^*$.

Lemma 2.3.4 *If $(Y, M_{|Y})$ has property HL, then (X, M) has properties HL_r for $r = 0$ and $2 \leq r \leq n$.*
The pair (X, M) has property HL_1 if and only if the restriction of the intersection form on $H^{n-1}(Y)$ to the subspace $i^*H^{n-1}(X)$ is non degenerate.
If $(Y, M_{|Y})$ has property HL, then (X, M) has property HL if and only if the restriction of the intersection form on $H^{n-1}(Y)$ to the subspace $i^*H^{n-1}(X)$ is non degenerate.

Proof. The case $r = 0$ is trivially true. Let $2 \leq r \leq n$. The first assertion follows from Proposition 2.1.5, the assumption that property HL holds for $(Y, M_{|Y})$ and Lemma 2.3.3. Let us prove the second statement. The map $L_M : H^{n-1}(X) \longrightarrow H^{n+1}(X)$ is the composition of the injective map $i^* : H^{n-1}(X) \longrightarrow H^{n-1}(Y)$ and its surjective transpose $i_* : H^{n-1}(Y) \longrightarrow H^{n+1}(X)$. It is an isomorphism iff $i^*H^{n-1}(X) \cap Ker\, i_* = \{0\}$, i.e. iff $i^*H^{n-1}(X) \cap (i^*H^{n-1}(X))^\perp = \{0\}$. This latter statement is equivalent to the non-degeneracy of the restriction of the intersection form. The third assertion follows. $\square$

We now prove Theorem 2.3.1. By Lemma 2.3.4 it is enough to show the statement of non-degeneracy on $i^*H^{n-1}(X)$. Conisder $i_* : H^{n-1}(Y) \to H^{n+1}(X)$. We have

$(i^*H^{n-1}(X))^\perp = Ker\, i_* \subseteq P^{n-1}_{M_{|Y}}(Y)$. By induction this last space is polarized by the intersection form. In particular, the intersection form in non-degenerate on $(i^*H^{n-1}(X))^\perp = Ker\, i_*$ so that it is non-degenerate on $i^*H^{n-1}(X)$. It follows that (X, M) has property HL. We conclude by Proposition 2.2.6. $\square$

2.3.2 The use of pencils

In this section we place ourselves in the same situation as in §2.3.1. The goal is to prove Theorem 2.3.10.

Choose a pencil $l \subseteq |M|$ which meets the set W of Proposition 2.1.7. There are two elements u_0 and $u_\infty \in l \cap W$ such that the corresponding nonsingular divisors meet transversally along a nonsingular subvariety A. By blowing up X along A we obtain a nonsingular variety X' together with a proper and flat morphism $f : X' \to l \simeq \mathbb{P}^1$. Denote by U the Zariski open and dense subset $l \cap W$. The morphism f is smooth over U. For every $u \in U$, the fiber $f^{-1}(u)$ is naturally isomorphic to the divisor Y_u corresponding to $u \in U \subseteq l$ and, in particular, $M_{|Y_u}$ is lef for every $u \in U$. We denote Y_{u_0} simply by Y and we denote the strict transform of Y on X' by Y'. Clearly, Y and Y' are naturally isomorphic. Let $j : U \to \mathbb{P}^1$ denote the open imbedding. The sheaves $j^*R^qf_*\mathbb{Q}_{X'}$ are local systems on U, for every $q \geq 0$, corresponding to the $\pi_1(U, u_0)$-module $H^q(Y')$. By abuse of notation, we drop the base point u_0.

The following is well-known, e.g. [15], Lemma 5.3:

Lemma 2.3.5 *Let $\mathcal{V}$ be local system on U associated with a finite dimensional $\pi_1(U)$-module V.*
We have that $H^0(\mathbb{P}^1, j_*\mathcal{V}) = H^0(U, \mathcal{V}) = V^{\pi_1(U)}$, the submodule of invariants.

We now invoke the following well-known result of Deligne's

Proposition 2.3.6 *Let $i' : Y' \longrightarrow f^{-1}(U)$ and $j' : f^{-1}(U) \longrightarrow X'$ be the corresponding closed and open imbeddings. For every q we have*

$$i'^*H^q(f^{-1}(U)) = (j' \cdot i')^*H^q(X') \subseteq H^q(Y').$$

Proof. See [7], Corollaire 3.2.18. $\square$

In what follows we do not want to use Theorem 2.3.1, so that our study will also yield a second proof of that result.

Proposition 2.3.7 *Assume that Theorem 2.3.1 holds in dimension $n-1$.*
The Leray spectral sequence for the restriction of f to $f^{-1}(U)$ degenerates at E_2. In particular, the morphism

$$H^q(f^{-1}(U)) \longrightarrow H^0(U, R^qf_*\mathbb{Q}_{X'}) = H^q(Y)^{\pi_1(U)}$$

is surjective for every q.

Proof. By virtue of the assumptions and of the choice of U, we can apply the degeneration criterion in [6]. □

Corollary 2.3.8 *Assume that Theorem 2.3.1 holds in dimension $n-1$. The image of $H^q(X')$ in $H^q(Y')$ by the restriction map is $H^q(Y')^{\pi_1(U)}$ for every q.*

Proof. The restriction map

$$i'^* : H^q(X') \longrightarrow H^q(Y')$$

factors as follows

$$H^q(X') \longrightarrow H^q(f^{-1}(U)) \longrightarrow H^0(U, R^q f_* \mathbb{Q}) \longrightarrow H^q(Y')^{\pi_1(U)} \longrightarrow H^q(Y').$$

The statement follows from Proposition 2.3.6 and Proposition 2.3.7. □

We now compare $H^q(X) \longrightarrow H^q(Y)$ with $H^q(X') \longrightarrow H^q(Y')$.

Proposition 2.3.9 *Let $i' : Y' \longrightarrow X'$ and $Bl_A : X' \longrightarrow X$ the blowing up map. We have that $i'^* \cdot Bl_A^*$ is injective. In addition, if $q \leq n-1$, then $i'^* H^q(X') = i'^* \cdot Bl_A^* H^q(X) \subseteq H^q(Y')$.*

Proof. See Éxposé XVIII Corollaire 5.1.6 in SGA VII,2. □

Again, we do not want to use Theorem 2.3.1. Recall that Y and Y' are naturally identified.

Theorem 2.3.10 *Assume that Theorem 2.3.1 holds in dimension $n-1$. For $q \leq \dim X - 1$ the map $i'^* \cdot Bl_A^* : H^q(X) \longrightarrow H^q(Y')$ is injective and its image is precisely the subspace $H^q(Y')^{\pi_1(U)}$ of monodromy invariants. In addition, the $\pi_1(U)$-representation on $H^{n-1}(Y')$ is semisimple.*

Proof. We only need to prove the last statement. The restriction of $Bl_A^* M$ to $f^{-1}(u)$ is lef for every $u \in U$. By virtue of the assumptions and of Theorem 2.2.6, this defines a polarization on all the primitive subspaces $P_M^q(u) \subseteq H^q(f^{-1}(u))$. It follows that $H^{n-1}(f^{-1}(u)) = \oplus L_M^i P_M^{n-1-2i}(u)$. The local systems $P_M^{n-1-2i}(u)$ define a variation of polarized Hodge structures. We conclude by the semisimplicity theorem of Deligne [7], Theorem 4.2.6 and footnote 1. □

2.3.3 A second proof of Theorem 2.3.1

We prove the theorem by induction on $\dim X = n$. The statement is true when $\dim X = 1$, for then M is necessarily ample.

Let us assume that the statement is true in dimension $n-1$. We replace, without loss of generality, M by a positive power which is generated by its global sections.

We fix a pencil l as in §2.3.2. By virtue of Theorem 2.3.10, we can find a $\pi_1(U)$-invariant complement V to $H^{n-1}(Y')^{\pi_1(U)}$ in $H^{n-1}(Y')$. The Poincaré duality pairing on $H^{n-1}(Y')$ is $\pi_1(U)$-invariant, therefore it restricts to a non-degenerate pairing on $H^{n-1}(Y')^{\pi_1(U)}$.

We conclude that (X, M) has the property HL by virtue of the third part of Proposition 2.3.4. Finally, we conclude by Proposition 2.2.6. □

2.4 The Hodge Index Theorem for semismall maps

Let us record the following consequence of Theorem 2.3.10 and of Proposition 2.1.6. Together with Remark 2.4.4, they are a higher dimensional analogue of Grauert contractibility test for curves on surfaces.

Theorem 2.4.1 *(Hodge Index Theorem for semismall maps) Let $f : X \to Y$ be a semismall map from a nonsingular complex projective variety of even dimension n onto a projective variety Y and $y \in Y$ be a point such that $\dim f^{-1}(y) = \frac{n}{2}$. Denote by Z_l, $1 \leq l \leq r$, the irreducible components of maximal dimension of $f^{-1}(y)$.*
Then the cohomology classes $[Z_l] \in H^n(X)$ are linearly independent and the symmetric matrix $(-1)^{\frac{n}{2}}||Z_l \cdot Z_m||$ is positive definite.

Proof. Choose a general pencil l of divisors on X as in §2.3.2 with the additional property that its general members satisfy the conclusion of Proposition 2.1.6 for $X \setminus f^{-1}(y)$. Let $V \subseteq l$ be the Zariski dense open subset such that the members X_v are nonsingular, $X_v \cap f^{-1}(v) = \emptyset$ and Proposition 2.1.6 holds for $X \setminus f^{-1}(y)$ and X_v. We have that the restriction map $H^{n-1}(X \setminus f^{-1}(y)) \to H^{n-1}(X_v)$ is injective for $v \in V$. By virtue of Theorem 2.3.10, we have the isomorphism $H^{n-1}(X) \to H^{n-1}(X_v)^{\pi_1(V)}$, for every $v \in V$. This isomorphism factors as follows

$$H^{n-1}(X) \to H^{n-1}(X \setminus f^{-1}(y)) \xrightarrow{a} H^{n-1}(X_v)^{\pi_1(V)} \subseteq H^{n-1}(X_v).$$

Hence the injective map a is surjective.
It follows that the natural cycle class map $H_n^{BM}(f^{-1}(y)) \simeq H^n(X, X \setminus f^{-1}(y)) \to H^n(X)$ is injective. Its image is contained in the primitive space associated with any line bundle on X pull-back of an ample line bundle on Y. The statement follows from Theorem 2.3.1. □

Corollary 2.4.2 *Let $f : X \to Y$ and y be as above. The mixed Hodge Structure on $H^{n-1}(X \setminus f^{-1}(y))$ is pure of weight $n-1$.*

Proof. It follows from the surjectivity of the natural map $H^{n-1}(X) \to H^{n-1}(X \setminus f^{-1}(y))$. □

Corollary 2.4.3 *Let $f : X \to Y$ be a birational semismall map from a nonsingular quasi projective complex variety of even dimension n onto a quasi projective complex variety with an isolated singularity $y \in Y$ such that f is an isomorphism outside y and $\dim f^{-1}(y) = \frac{n}{2}$. Then the conclusions of Theorem 2.4.1 hold.*

Proof. One finds a semismall projective completion $f'' : X'' \to Y''$ of f to which we apply Theorem 2.4.1. Since the bilinear form on the fibers in non-degenerate, the cycle classes of the fundamental classes of the fibers stay independent in $H^n(X)$. □

Remark 2.4.4 The same proof shows that the statements of Theorem 2.4.1 and Corollary 2.4.3 holds unchanged if we consider any finite number of fibers over points as y above.

Remark 2.4.5 If (Y, y) is a germ of a normal complex space of dimension two, and $f : X \to Y$ is a resolution of singularities, then Grauert Contractibility Criterion, see [14] Theorem 4.4, implies that the form in question is negative definite and, in particular, it is non-degenerate.

The following is a natural question. A positive answer would yield a proof of the Decomposition Theorem for semismall holomorphic maps from complex manifolds and for constant coefficients; see Theorem 3.3.3.

Question 2.4.6 Let $f : V \to W$ be a proper holomorphic semismall map from a complex manifold of even dimension n onto an analytic space Y. Assume that the fiber $f^{-1}(w)$ over a point $w \in W$ has dimension $\frac{n}{2}$. Is the intersection form on $H_n^{BM}(f^{-1}(w))$ non-degenerate? Is it $(-1)^{\frac{n}{2}}$-positive definite?

3 The topology of semismall maps

We now proceed to a study of holomorphic semismall maps from a complex manifold. First we need to prove Proposition 3.1.2, a simple splitting criterion in derived categories for which we could not find a reference. We study the topology of these maps by attaching one stratum at the time. In doing so a symmetric bilinear form emerges naturally; see Proposition 3.2.4 and Lemma 3.2.5. We then prove that the Decomposition Theorem for these maps and for constant coefficients is equivalent to the non-degeneration of these forms; Theorem 3.3.3. Finally, we give a proof of the Decomposition Theorem when the domain and target are projective, Theorem 3.4.1. A new feature that we discover is that the forms are definite by virtue of our Hodge Index Theorem for semismall maps.

3.1 Homological algebra

Let $\mathcal{A}$ be an abelian category with enough injectives, e.g. sheaves of abelian groups on a topological space, and $C(\mathcal{A})$ be the associated category of complexes. Complexes and morphisms can be truncated. Given an integer t, we have two types of truncations: $\tau_{\leq t}A$ and $\tau_{\geq t}A$. The former is defined as follows $(\tau_{\leq t}A)^i := A^i$ for $i \leq t-1$, $(\tau_{\leq t}A)^t := Ker(A^t \to A^{t+1})$, $(\tau_{\leq t}A)^i := \{0\}$ for $i > t$. The latter is defined as follows $(\tau_{\geq t}A)^i := \{0\}$ for $i \leq t-1$, $(\tau_{\geq t}A)^t := Coker(A^{t-1} \to A^t)$, $(\tau_{\geq t}A)^i := A^i$ for $i > t$. Let $h : A \to B$ be a morphism of complexes. The truncations $\tau_{\leq t}(h) : \tau_{\leq t}A \to \tau_{\leq t}B$ and $\tau_{\geq t}(h) : \tau_{\geq t}A \to \tau_{\geq t}B$ are defined in the natural way. The operations of truncating complexes and morphisms of complexes induce functors in the derived category $D(\mathcal{A})$.

If A is a complex acyclic in degrees $l \neq t$ for some integer t, i.e. if $\tau_{\leq t}A \simeq \tau_{\geq t}A$, then $A \simeq \mathcal{H}^t(A)[-t]$.

The cone construction for a morphism of complexes $h : A \to B$ gives rise, in a non-unique way, to a diagram of morphism of complexes $A \stackrel{h}{\to} B \to M(h) \stackrel{[1]}{\to} A[1]$. A diagram of morphisms $X \to Y \to Z \stackrel{[1]}{\to} X[1]$ in $D(\mathcal{A})$ is called a distinguished triangle if it is isomorphic to a diagram arising from a cone.

A morphism $h : A \to B$ in $D(\mathcal{A})$ gives rise to a distinguished triangle $A \stackrel{h}{\to} B \to C \to A[1]$. If $h = 0$, then $C \simeq A[1] \oplus B$ and the induced morphism $A[1] \to A[1]$ is an isomorphism.

A morphism $h : A \to B$ in the derived category gives a collection of morphisms in cohomology $\mathcal{H}^l(h) : \mathcal{H}^l(A) \to \mathcal{H}^l(B)$. A distinguished triangle $A \to B \to C \stackrel{[1]}{\to} A[1]$ gives rise to a cohomology long exact sequence:

$$\ldots\ \mathcal{H}^l(A) \to \mathcal{H}^l(B) \to \mathcal{H}^l(C) \to \mathcal{H}^{l+1}(A)\ \ldots$$

A non-zero morphism $h : A \to B$ in the derived category may nonetheless induce the zero morphisms between all cohomology groups. However, we have the following simple and standard

Lemma 3.1.1 *Let t be an integer and A and B be two complexes such that $A \simeq \tau_{\leq t}A$ and $B \simeq \tau_{\geq t}B$. Then the natural map $Hom_{D(\mathcal{A})}(A, B) \to Hom_{\mathcal{A}}(\mathcal{H}^t(A), \mathcal{H}^t(B))$ is an isomorphism of abelian groups.*

Proof. It is enough to replace B by an injective resolution placed in degrees no less than t. □

We shall need the following elementary splitting criterion.

Proposition 3.1.2 *Let $C \xrightarrow{u} A \xrightarrow{v} B \xrightarrow{[1]} C[1]$ be a distinguished triangle and t be an integer such that $A \simeq \tau_{\leq t}A$ and $C \simeq \tau_{\geq t}C$.*
Then $\mathcal{H}^t(u) : \mathcal{H}^t(C) \to \mathcal{H}^t(A)$ is an isomorphism iff

$$A \simeq \tau_{\leq t-1}B \,\oplus\, \mathcal{H}^t(A)[-t]$$

and the map v is the direct sum of the natural map $\tau_{\leq t-1}B \to B$ and the zero map.

Proof. Assume that $\mathcal{H}^t(u)$ is an isomorphism. Apply the functor $Hom(A, -)$ to the distinguished triangle $\tau_{\leq t-1}B \xrightarrow{\nu_{t-1}} \tau_{\leq t}B \xrightarrow{\pi} \mathcal{H}^t(B)[-t] \xrightarrow{[1]} \tau_{\leq t-1}B\,[1]$ and we get the following exact sequence:

$$\ldots \to Hom^{-1}(\tau_{\leq t}A, \mathcal{H}^t(B)[-t]) \to Hom^0(\tau_{\leq t}A, \tau_{\leq t-1}B) \to$$

$$Hom^0(\tau_{\leq t}A, \tau_{\leq}B) \to Hom^0(\tau_{\leq t}A, \mathcal{H}^t(B)[-t])) \to \ldots$$

Since $\mathcal{H}^t(B)[-t]$ is concentrated in degree t, $Hom^{-1}(\tau_{\leq t}A, \mathcal{H}^t(B)[-t]) = \{0\}$. The morphism $\mathcal{H}^t(v) = 0$, for $\mathcal{H}^t(u)$ is surjective.

It follows that there exist a unique lifting v' of $\tau_{\leq t}(v)$, i.e. there exists a unique $v' : A \to \tau_{\leq t-1}B$ such that $\tau_{\leq t}(v) = \nu_{t-1} \circ v'$.
We complete v' to a distinguished triangle:

$$\tau_{\leq t}A \xrightarrow{v'} \tau_{\leq t-1}B \xrightarrow{v''} M(v') \xrightarrow{[1]} \tau_{\leq t}A\,[1]\ .$$

By degree considerations, the morphism $\mathcal{H}^l(v') = 0$ for $l \geq t$. Since v' is a lifting of $\tau_{\leq t}(v)$, the morphism $\mathcal{H}^l(v')$ is an isomorphism for $l \leq t-1$ and it is the zero map for $l \geq t$. This implies that $M(v') \simeq \mathcal{H}^t(A)[-t+1]$ and that $\mathcal{H}^{t-1}(v'') = 0$. By virtue of Lemma 3.1.1, we get that $v'' = 0$.
The desired splitting follows. The converse can be read off the long exact cohomology sequence. $\square$

3.2 The bilinear forms associated with relevant strata

Let $f : X \to Y$ be a proper holomorphic semismall map with X nonsingular connected of dimension n. Let us summarize the results from stratification theory (cf. [12], Ch. 1) that we shall need in the sequel. They are based essentially on Thom First Isotopy Lemma.

There exists a collection of disjoint locally closed and *connected* analytic subvarieties $Y_i \subseteq Y$ such that:

a) $Y = \coprod_i Y_i$ is a Whitney stratification of Y.
b) $Y_i \cap \overline{Y_j} \neq \emptyset$ iff $Y_i \subseteq \overline{Y_j}$.
c) the induced maps $f_i : f^{-1}(Y_i) \to Y_i$ are stratified submersions; in particular they are topologically locally trivial fibrations.

We call such data *a stratification of the map f*.

Definition 3.2.1 A stratum Y_i is said to be *relevant* if $2\dim f^{-1}(Y_i) - \dim Y_i = n$. Let $I' \subseteq I$ be the set of indices labeling relevant strata.

Let $i \in I$ be any index and $d_i := \dim Y_i$. Define $\mathcal{L}_i := (R^{n-d_i} f_* \mathbb{Q}_X)_{|Y_i}$. It is a local system on Y_i.

Remark 3.2.2 If Y_i is not relevant, then $\mathcal{L}_i$ is the zero sheaf. If Y_i is relevant, then the stalks $(\mathcal{L}_i^*)_{y_i} \simeq H^{BM}_{n-d_i}(f^{-1}(y_i))$ of the dual local system are generated exactly by the fundamental classes of the irreducible and reduced components of maximal dimension of the fiber over y_i.

The following is elementary and holds also when the stratum is not relevant when we consider the local system dual to the one generated by the components of maximal dimension of the fibers.

Lemma 3.2.3 *The local system $\mathcal{L}_i$ splits as a direct sum $\mathcal{L}_i \simeq \bigoplus_{j=1}^{m_i} \mathcal{L}_{ij}$ of irreducible local sub-systems.*

Proof. It is enough to show the statement for $\mathcal{L}_i^*$, $i \in I'$. Going around a loop in Y_i has the effect of permuting the elements of the basis of Remark 3.2.2. The associated monodromy representation factors through a finite symmetric group so that it splits into a sum of irreducibles. $\square$

Let $S := Y_i$, $d := \dim S$ and $\mathcal{L}_S := \mathcal{L}_i$. We now proceed to associating with S a symmetric bilinear form on the local system $\mathcal{L}_S^*$.

Let $s \in S$ and choose a small-enough euclidean neighborhood U of s in Y such that a) $S' := S \cap U$ is contractible and b) the restriction $i^* : H^{n-d}(f^{-1}(U)) \to H^{n-d}(f^{-1}(s))$ is an isomorphism.

Let $F_1, \ldots, F_r$ be the irreducible and reduced components of maximal dimension of $f^{-1}(S')$. By virtue of a) above and of the topological triviality over S', the intersections $f_j := f^{-1}(s) \cap F_j$ are exactly the irreducible and reduced components of maximal dimension of $f^{-1}(s)$. The analogous statement is true for every point $s' \in S'$ and the components for the point s can be canonically identified with the ones of s'. The specialization morphism $i_s^! : H_{n+d}^{BM}(f^{-1}(S')) \to H_{n-d}^{BM}(f^{-1}(s))$, associated with the regular imbedding $i_s : \{s\} \to S'$, sends the fundamental class of a component F_l to the fundamental class of the corresponding f_l and it is an isomorphism; see [10], Ch. 10. We have the following sequence of maps:

$$H_{n-d}^{BM}(f^{-1}(s)) \xrightarrow{(i_s^!)^{-1}} H_{n+d}^{BM}(f^{-1}(S')) \xrightarrow{(\cap\mu_{f^{-1}(U)})^{-1}} H^{n-d}(f^{-1}(U), f^{-1}(U \setminus S'))$$

$$\xrightarrow{nat} H^{n-d}(f^{-1}(U)) \xrightarrow{i^*} H^{n-d}(f^{-1}(s)) \xrightarrow{\kappa} H_{n-d}^{BM}(f^{-1}(s))^*.$$

The second map is the inverse to the isomorphism given by capping with the fundamental class $\mu_{f^{-1}(U)}$ (cf. [13], IX.4). The third map is the natural map in relative cohomology. The fourth map is an isomorphism by virtue of condition b) above. The map κ is an isomorphism by the compactness of $f^{-1}(s)$.

We denote the composition, which is independent of the choice of U:

$$\rho_{S,s} : H_{n-d}^{BM}(f^{-1}(s)) \longrightarrow H_{n-d}^{BM}(f^{-1}(s))^*.$$

We have that

$$\rho_{S,s}(f_h)(f_k) = \deg F_h \cdot f_k,$$

where the refined intersection product takes place in $f^{-1}(U)$ and has values in $H_0^{BM}(f^{-1}(s))$.

Since the map f is locally topologically trivial along S, the maps $\rho_{S,s}$ define a map of local systems

$$\rho_S = \rho : \mathcal{L}_S^* \longrightarrow \mathcal{L}_S.$$

We record the following fact for future use.

Proposition 3.2.4 *If S is not relevant, then ρ_S is the zero map between trivial local systems. Let $s \in S$ be a point. The map $\rho_{S,s}$ is an isomorphism iff the natural map*

$r_k : H^k(f^{-1}(U)) \to H^k(f^{-1}(U \setminus S'))$ *is an isomorphism for every* $k \leq n - d - 1$*, iff the natural map* $s_k : H^k(f^{-1}(U \setminus S')) \to H^{k+1}(f^{-1}(U), f^{-1}(U \setminus S'))$ *is an isomorphism for every* $k \geq n - d$.

Proof. The domain and the range of $\rho_{S,s}$ are dual to each other. The statement follows from the relative cohomology sequence for the pair $(f^{-1}(U), f^{-1}(U \setminus S'))$, the isomorphisms $H^k(f^{-1}(U)) \simeq H^k(f^{-1}(s))$, $H^k(f^{-1}(U), f^{-1}(U \setminus S')) \simeq H^{BM}_{2n-k}(f^{-1}(S'))$ and the fact that $\dim f^{-1}(s) \leq \frac{n-d}{2}$, $\dim f^{-1}(S) \leq \frac{n+d}{2}$. $\square$

Since $f_i : f^{-1}(S) \to S$ is a stratified submersion, given any point $s \in S$, we can choose an analytic normal slice $N(s)$ to S at s such that $f^{-1}(N(s))$ is a locally closed complex submanifold of X of dimension $n - d$. We now use this fact to express the map $\rho_{S,s}$ in terms of the refined intersection pairing on $f^{-1}(N(s))$.

Lemma 3.2.5 *If* $s \in S$*, then* $\rho_{S,s}(f_h)(f_k) = \deg f_h \cdot f_k$*, where the refined intersection product on the r.h.s. takes place in* $f^{-1}(N(s))$ *and has values in* $H_0^{BM}(f^{-1}(s))$*. In particular, the map* $\rho_S : \mathcal{L}_S^* \longrightarrow \mathcal{L}_S$ *is symmetric.*

Proof. Since $f_i : f^{-1}(S) \to S$ is a stratified submersion and $N(s)$ is a normal slice to S at s, F_j meets $f^{-1}(N(s))$ transversally at the general point of f_j. It follows that the refined intersection product $f^{-1}(N(s)) \cdot F_j$ is the fundamental class of f_j in $H^{BM}_{n-d}(f^{-1}(s))$. The result follows by applying [10], 8.1.1.a) to the maps $f^{-1}(s) \to f^{-1}(N(s)) \to f^{-1}(U)$. $\square$

3.3 Inductive study of semismall analytic maps

Let $f : X \to Y$ and $\{Y_j\}$, $i \in I$, S be as in section 3.2. We assume, for simplicity, the Y_i to be connected and I to be finite. There is no loss of generality, for strata of the same dimension do not interfere with each other from the point of view of the analysis that follows and could be treated simultaneously. As usual, we define a partial order on the index set I by setting $i \prec j$ iff $Y_i \subseteq \overline{Y_j}$. We fix a total order $I = \{i_1 < \ldots < i_\iota\}$ which is compatible with the aforementioned partial order and define the open sets $U_{\geq i} := \coprod_{j \geq i} Y_j$. Similarly, $U_{>i} := \coprod_{j>i} Y_j$. Let $\alpha_i : U_{>i} \to U_{\geq i}$ be the open imbedding. We can define the intermediate extension of a complex of sheaves $K^\bullet$ on $U_{>i}$ to a complex of sheaves on $U_{\geq i}$ by setting

$$\alpha_{i!*}K^\bullet = \tau_{\leq -dimY_i - 1} R\alpha_{i*}K^\bullet.$$

See [1]. The construction is general and can be iterated so that one can form the intermediate extension of a complex of sheaves on any Y_i to a complex on $\overline{Y_i} \cap U_{>j}$ for $j < i$. In particular, let $\mathcal{L}$ be a local system on Y_i. The intermediate extension of $\mathcal{L}[\dim Y_i]$ to $\overline{Y}_i \cap U_{>j}$ for $j < i$ is called the *intersection cohomology complex* associated with $\mathcal{L}$ and is denoted by $IC_{\overline{Y}_i \cap U_{>j}}(\mathcal{L})$.

Definition 3.3.1 Let $f : X \to Y$ be a proper holomorphic semismall map from a non-singular connected complex manifold X of dimension n. We say that *the Decomposition Theorem holds for f* if there is an isomorphism

$$Rf_*\mathbb{Q}_X[n] \simeq \bigoplus_{k\in I'} IC_{\overline{Y_k}}(\mathcal{L}_k) \simeq \bigoplus_{k\in I'}\bigoplus_{m=1}^{m_k} IC_{\overline{Y_k}}(\mathcal{L}_{km}),$$

where the $\mathcal{L}_{km}$ are as in Lemma 3.2.3.

Remark 3.3.2 The Decomposition Theorem holds, in the sense defined above, for X, Y and f algebraic (cf. [1]) and for f a Kähler morphism (cf. [17]). In both cases, a far more general statement holds. As observed in [3], §1.7, in the case of semismall maps these results can be expressed in the convenient form of Definition 3.3.1.

We now proceed to show that the non-degeneracy of the forms ρ_S associated with the strata Y_i implies the Decomposition Theorem.

Recall that $I' \subseteq I$ is the subset labeling relevant strata. For ease of notation set

$$V := U_{>i}, \qquad V' := U_{\geq i}, \qquad S := Y_i$$

and let

$$V \xrightarrow{\alpha} V' \xleftarrow{\beta} S$$

be the corresponding open and closed imbeddings.

Theorem 3.3.3 *Assume that the Decomposition Theorem holds over V. The map ρ_S is an isomorphism iff the Decomposition Theorem holds over V' and the corresponding isomorphism restricts to the given one over V.*

Proof. Denote by g the map $f_| : f^{-1}(V) \to V$. By cohomology and base change, $(Rf_*\mathbb{Q}_X[n])_{|V} \simeq Rg_*\mathbb{Q}_{f^{-1}(V)}[n]$. Similarly for V'. Clearly, we have $(Rf_*\mathbb{Q}_X[n])_{|V} \simeq \alpha^*[(Rf_*\mathbb{Q}_X[n])_{|V'}]$.
There is a distinguished "attaching" triangle, see [2], 5.14:

$$\beta_*\beta^!(Rf_*\mathbb{Q}_X[n]_{|V'}) \xrightarrow{u} (Rf_*\mathbb{Q}_X[n])_{|V'} \xrightarrow{v} R\alpha_*(Rf_*\mathbb{Q}_X[n]_{|V}) \xrightarrow{w\ [1]} \beta_*\beta^!(Rf_*\mathbb{Q}_X[n]_{|V'})[1].$$

On the open set V the complex $\beta_*\beta^!(Rf_*\mathbb{Q}_X[n])_{|V'}$ is isomorphic to zero and the map v restricts to an isomorphism. Recalling the notation in §3.2, the long exact sequence of cohomology sheaves is, stal-kwise along the points of S, the long exact sequence for the cohomology of the pair $(f^{-1}(U), f^{-1}(U \setminus S'))$. In addition the map $\mathcal{H}^{-d}(u)$ is identified, stalk-wise along the points of S, with the map $\rho_{S,s}$. The statement follows from Proposition 3.2.4 which allows us to apply proposition 3.1.2. □

Remark 3.3.4 In the algebraic and Kähler case, the results [1] and [17], coupled with Theorem 3.3.3, imply that the forms ρ_S are non-degenerate for every $i \in I$; see also [4], Theorem 8.9.14. To our knowledge these results have no implications as to the sign of the intersection forms. Surprisingly, in the projective case we can determine that these forms are definite; see §3.4.

The following statement is known as a consequence of the characterizing properties of intersection cohomology sheaves; see [11], §6.2 and [3], §1.8. Recall that a map is *small* if all the inequalities in the definition of semismall are strict. Note that in this case there is only one relevant stratum.

Corollary 3.3.5 *Let $f : X \to Y$ be a proper holomorphic small map from a connected complex manifold. Then the Decomposition Theorem holds for f.*

Proof. All the maps ρ_S are identically zero and are isomorphisms. □

3.4 Signature and Decomposition Theorem in the projective case

In this section we use Theorem 2.4.1, the previous inductive analysis and a Bertini-type argument to give a proof of the following theorem

Theorem 3.4.1 *Let $f : X \to Y$ be a semismall map from a nonsingular complex projective variety of dimension n onto a complex projective variety. The Decomposition Theorem holds for f, i.e. there is a canonical isomorphism*

$$Rf_*\mathbb{Q}_X[n] \simeq \bigoplus_{k\in I'} IC_{\overline{Y_k}}(\mathcal{L}_k) \simeq \bigoplus_{k\in I'} \bigoplus_{m=1}^{m_k} IC_{\overline{Y_k}}(\mathcal{L}_{km}).$$

For every relevant stratum S of dimension d the associated intersection form is non-degenerate and $(-1)^{\frac{n-d}{2}}$-definite.

Proof. By virtue of Proposition 3.3.3 we are reduced to checking that the intersection form associated with a relevant d-dimensional stratum S is non-degenerate and $(-1)^{\frac{n-d}{2}}$-definite.
If $d = 0$, then the conclusion follows from Proposition 2.4.1.
Let $d > 0$. Let A be a very ample divisor on Y. The line bundle $M := f^*A$ is lef and generated by its global sections. By virtue of Proposition 2.1.7, we can choose d general sections $H_1, \ldots, H_d$ in the linear system $|A|$ such that their common zero locus H has the property that $f^{-1}(H)$ is nonsingular of dimension $n - d$, $f^{-1}(H) \to H$ is semismall, H meets S at a non-empty finite set of points $s_1, \ldots, s_r$ so that, for at least one index $1 \le l \le r$, a small neighborhood of a point s_l in H is a normal slice to S at s_l. By virtue of Theorem 2.4.1 the intersection form of $f^{-1}(s_l) \subseteq f^{-1}(H)$ has the required properties at the point s_l, and therefore at every point $s \in S$. We conclude by applying Lemma 3.2.5. □

Remark 3.4.2 Theorem 3.4.1 can be applied even when the spaces are not complete, in the presence of a suitable completion of the morphism: one for which the domain is completed to a projective manifold, the target to a projective variety and the map to a semismall one. In general this may not be possible, but it can be done in several instances, e.g. the Springer resolution of the nilpotent cone of a complex semisimple Lie algebra, the Hilbert scheme of points on an algebraic surface mapping on the corresponding symmetric product, isolated singularities (see below), certain contraction of holomorphic symplectic varieties ...

Corollary 3.4.3 *Let $f : X \to Y$ be a birational semismall map from a nonsingular quasi projective complex variety of dimension n onto a quasi projective complex variety Y with isolated singularities. Assume that f is an isomorphism outside the isolated singularities. The Decomposition Theorem holds for f.*

Proof. We can reduce the statement to the complete projective case: see Corollary 2.4.3 and Remark 2.4.4. □

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